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Population Dynamical Model for AIDS Patients of a Particular Area

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Abstract. This is an attempt to translate the problem of AIDS patients into a mathematical problem, thereafter interpreting the solution in the language of real world. Here we proposed HIV/AIDS among the peoples as a resultant function of their unnatural sexual intercourses and obtained exponential HIV/AIDS patients growth model and logistic curve for HIV/AIDS patients. The interpretation of the logistic curve showed that the HIV/AIDS patient growth is massive and needs urgent care in terms of personal vulnerability of HIV infections.

Key words: AIDS Patient, Mathematical Problem, Resultant function, Unnatural Sexual Intercourses Exponential HIV/AIDS Patients Growth Model and Logistic Curve.
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1. Introduction

AIDS is incurable disease that slowly attacks and destroys the body's immune system. AIDS (Acquired Immune Deficiency Syndrome) is not hereditary and is characterized by a number of symptoms occurring together. The term syndrome is therefore used for defining AIDS. It is the HIV i.e. the Human Immune Deficiency Virus that finally leads to AIDS. The presence of HIV is particularly high in blood, semen of man, cerebrospinal fluid, and vaginal and cervical secretions of the woman. The HIV is transmitted through unnatnral sexual intercourse, hetero sexnal or homosexual, either vaginal sex, oral sex or anal sex.
The statistics regarding HIV/AIDS given paper (Table 1) are those from United Nations Programme on HIV/AIDS (UNAIDS) and the World Health Organization (WHO) publications [2],[3], [4], [5].From data it is observed that the total

number of AIDS patients was 33.4 million, and the total number of AIDS deaths was 2.5 million in 1998, and 39.5 million AIDS patients and the total number of AIDS deaths was 2.9 million in 2006. The people who had high-risk behavior for HIV infection were due to mainly MSM (Men having sex with men), FSW (female sex workers) and unnatural hetero sexuality. Unnatural sexual (hetero or homo sexual) promiscuous behaviour is the most probable source of infection. More than 80 percent AIDS patients are due to unnatural hetero sexual or homo sexual relations ([6], [7], [8], [9]). AIDS patients in India were reported 86 percent in 2005 due to hetero sexual or homo sexual promiscuous behaviour [10].

Hypothesis: If the situation modeled some continuous variable(s) and we have some reasonable hypothesis about the rates of change of dependent variable(s); mathematical modeling in terms of differential equations arises. When we have a variable x depending on an independent variable t , we obtain a mathematical model in terms of ordinary differential equation of the first order if the hypothesis is about the rate of change $\frac{dx}{dt}$ [1].

Since the dynamics of HIV/AIDS infected population a particular area is depends on social and economical saturations which generate the factors responsible for the development and transmission of HIV. Let x be a parameter on which the dynamics of number of HIV/AIDS infected population depend and composed of two factors say α , which denotes the rate of increase, and β denote the rate of retardation of HIV/AIDS infected population

Then the total A be the number of AIDS patients of a particular area, then A will be the function of x i.e. $A(x)$. Then at $(x+\Delta x)$, number of AIDS patients will be equal to $A(x+\Delta x)$, and the change in HIV/AIDS infected population will be $A(x+\Delta x) - A(x)$ at the interval Δx . Then

$$A(x+\Delta x) - A(x) = (\alpha - \beta)A\Delta X$$

Suppose $\mu = \alpha - \beta$ Therefore

$$A(x+\Delta x) - A(x) = \mu A\Delta X$$

On dividing by Δx and taking $\Delta x \to 0$, we get

$$\frac{dA}{dx} = \mu A \tag{1}$$

Integrating equation (1), results

$$\log A = \mu x + \log C_1$$

$$A = C_1 e^{\mu x} \tag{2}$$

Now at $x = 0$, $A = A(0)$ which represents the value of A when the effect of the parameter x is insignificant, and then $C_1 = A(0)$, therefore

$$A(x) = A(0)e^{\mu x} \tag{3}$$

Which is an exponential curve ([11], [12]) .

Table 1. Regional HIV/ AIDS Statistics

Region	Adults & children living with HIV/AIDS			Adults & children newly infected with HIV			Adults & Child Deaths due to HIV			Mode(s) of transmission for adults living with HIV/ AIDS
	2003	2004	2005	2003	2004	2005	2003	2004	2005	
Sub Saharan Africa	24.9 M	25.4 M	25.8 M	3.0 M	3.1 M	3.2 M	2.1 M	2.3 M	2.4 M	Hetero, MSN
Middle East and North Africa	500000	540000	510000	62000	92000	67000	55000	28000	58000	Hetero, IDU
Asia	7.1 M	8.2 M	8.3 M	940000	1.18 M	1.1 M	420000	541000	520000	Hetero, Sex worker, IDU
Latin America	1.6 M	1.7 M	1.8 M	170000	240000	200000	59000	95000	66000	MSM, Hetero, IDU
Caribbean	300000	440000	300000	29000	53000	30000	24000	36000	24000	MSM, Hetero,
Eastern Europe and Central Asia	1.2 M	1.4 M	1.6 M	270000	210000	270000	36000	60000	62000	MSM, Hetero, IDU
North America, Western and Central Europe	1.8 M	1.16 M	1.9 M	63000	65000	65000	30000	32500	30000	MSM, Hetero, IDU
Oceania	63000	35000	74000	8900	5000	8200	2000	7000	3600	MSM, Hetero, IDU
Total (million)	37.463 M	39.325	40.284	4.5424	4.945	4.9102	2.726	3.0995	3.1636	

MSM = Men having sex with men, IDU= Injecting Drug Use, M= million

Source: (i) **UNAIDS and WHO, AIDS Epidemic Update: December 2004**
(ii) UNAIDS and WHO, AIDS Epidemic Update: December 2005

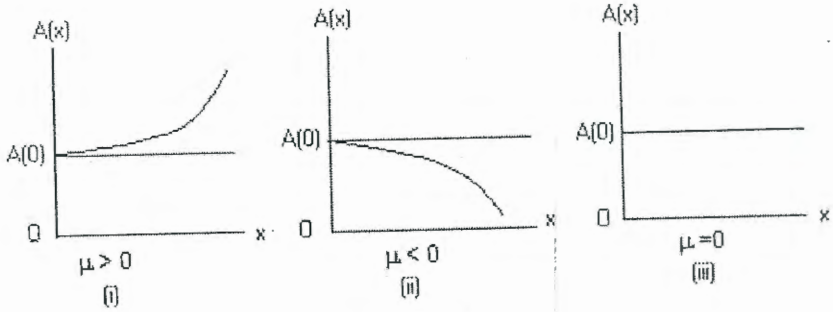

Figure 1

From Logistic Theory there may be many sub parameter which decrease or increase the population of HIV/AIDS patients then α and β may be further cauterized and can be given

$$\alpha = \alpha_1 - \alpha_2 A, \quad \beta = \beta_1 + \beta_2 A, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$$

Therefore

$$\alpha - \beta = (\alpha_1 - \beta_1) - (\alpha_2 + \beta_2)A = \mu - \gamma A$$

and equation (1) becomes

(4) $$\frac{dA}{dx} = (\mu - \gamma A)A \quad \mu > 0, \gamma > 0$$

On integration, we obtain

(5) $$\frac{A(x)}{\mu - \gamma A(x)} = C_2 e^{\mu x}$$

or

(6) $$A(x) = \frac{1}{\frac{\gamma}{\mu} + \frac{1}{C_2} e^{-\mu x}} = \frac{\mu/\gamma}{1 + \frac{1}{\gamma C_2} e^{-\mu x}}$$

Let λ be the value of x for which $A(x) = \mu/2\gamma$. Then

$$\frac{\mu}{2\lambda} = \frac{\mu/\gamma}{1 + \frac{1}{\gamma C_2} e^{-\mu\lambda}} \quad \Rightarrow C_2 = \frac{1}{\gamma} e^{-\mu\lambda}$$

Substituting this in equation (6)

(7) $$A(x) = \frac{\mu/\gamma}{1 + e^{\mu(\lambda - x)}} = \frac{L}{1 + e^{\mu(\lambda - x)}}$$

This is the form in which the equation of logistic curve is generally expressed. But

$$C_2 = \frac{A(0)}{\mu - \gamma A(0)} \quad for \quad x = 0$$

Therefore

$$(8) \qquad \frac{A(x)}{\mu - \gamma A(x)} = \frac{A(0)}{\mu - \gamma A(0)} e^{\mu x}$$

Form (4)

$$(9) \qquad \frac{d^2 A}{dx^2} = \mu - 2\gamma A$$

so that

$$(10) \qquad \frac{d^2 A}{dx^2} \stackrel{>}{<} 0 \qquad according \quad as \qquad A \stackrel{<}{>} \frac{L}{2} = \frac{\mu}{2\gamma}$$

The critical value $\frac{L}{2}$ occurs when $x = \lambda$. Thus the patients growth curve is convex if $A < \frac{L}{2} = \frac{\mu}{2\gamma}$ and concave if $A > \frac{L}{2} = \frac{\mu}{2\gamma}$ and it has a point of inflexion at $A = \frac{L}{2} = \frac{\mu}{2\gamma}$. Equations (4) and (8) show that

(i) $A(0) < L = \frac{\mu}{\gamma} \quad \Rightarrow A(x) < L = \frac{\mu}{\gamma} \quad \Rightarrow \frac{dA}{dx} > 0$

This implies that $A(x)$ is monotonic increasing function of x which approaches $L = \frac{\mu}{\gamma}$ as $x \to \infty$, L is called the saturation level of the patients.

(ii) $A(0) > \frac{\mu}{\gamma} \quad \Rightarrow A(x) > \frac{\mu}{\gamma} \quad \Rightarrow \frac{dA}{dx} < 0$

This implies that $A(x)$ is monotonic decreasing function of x which approaches $\frac{\mu}{\gamma}$ as $x \to \infty$.

(iii) The curve is skew symmetric in the sense that

$$A(\lambda) - A(\lambda - h) = A(\lambda + h) - A(\lambda)$$

$$= \frac{\mu}{2\gamma}\left[\frac{e^{\mu h} - 1}{e^{\mu h} + 1}\right] \qquad for \quad every \quad h$$

The above properties taken together indicate that the curve is shaped like an elongated S [13]. The graph of $A(x)$ against x is given below:

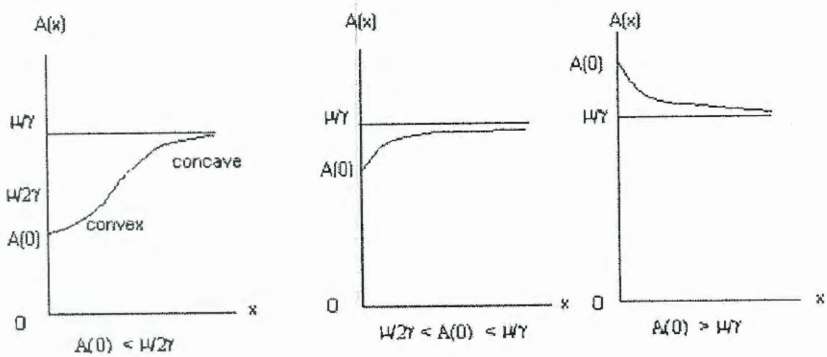

Figure 2

2. Fitting of a Logistic Curve (Method of Rhodes)

If the AIDS patients follows a logistic curve strictly,

$$\frac{1}{A(x-1)} = \frac{1}{L} + \frac{e^{\mu(\lambda-x+1)}}{L}, \quad \frac{1}{A(x)} = \frac{1}{L} + \frac{e^{\mu(\lambda-x)}}{L}$$

Therefore,

$$\frac{1}{A(x)} = \frac{1-e^{-\mu}}{L} + \frac{e^{-\mu}}{A(x-1)} \tag{11}$$

Writing

$$\frac{1}{A(x)} = y(x), \quad \frac{1}{A(x-1)} = X^{'}(x), \quad A = \frac{1-e^{-\mu}}{L}, \quad B = e^{-\mu} \tag{12}$$

Equation (11) can be express as

$$y(x) = A + BX^{'}(x) \tag{13}$$

Thus if the HIV/AIDS infected population follows exactly a logistic curve then $y(x)$ and $X^{'}(x)$ will be linearly related. However it is observed that HIV/AIDS infected population does not follow logistic exactly. Denoting $A^{'}(x)$ the observed HIV/AIDS infected population at parameter x and

$$Y(x) = \frac{1}{A'(x)}, \quad X^{'} = \frac{1}{X''(x)}$$

we shall have

$$y(x) = A + BX^{'}(x) + e(x)$$

where $e(x)$ is the error due to deviation of observed AIDS patients from the logistic patients.
The estimates are

$$B\wedge = b = \sqrt{\frac{\sum_1^{n-1}(Y(x)-\overline{Y})^2}{\sum_1^{n-1}(X(x)-\overline{X})^2}}; \qquad A\wedge = a = \overline{Y} - b\overline{X}$$

where

$$\overline{X} = \frac{1}{N-1}\sum_1^{N-1} X(x), \qquad \overline{Y} = \frac{1}{N-1}\sum_1^{N-1} Y(x) = \overline{X} + \frac{1}{N-1}[\frac{1}{A(N-1)} - \frac{1}{A(0)}]$$

L and μ are estimated from estimates of A and B by relations (12). Finally λ is estimated by noting that for the logistic curve,

$$\lambda = \frac{1}{\mu}\log(\frac{L}{A(x)} - 1) + x$$

An estimates of λ is obtained as the arithmetic mean

$$(14) \qquad \lambda^\wedge = \frac{1}{rN}\sum_{x=0}^{N-1}(\log(\frac{L}{A'(x)} - 1)) + \frac{N-1}{2}$$

Table 2. HIV/AIDS statistics and featuress: 1998-2006

Years	Adults & children living with HIV/AIDS (million)	Adults & children newly infected with HIV (million)	Main mode(s) of transmission for adults living with HIV/ AIDS
1998	33.40	5.800	Hetero, IDU, MSM, Sex workers
1999	34.132	5.610	"
2000	34.895	5.120	"
2001	35.612	4.900	"
2002	36.513	4.720	"
2003	37.463	4.542	"
2004	39.325	4.945	"
2005	40.284	4.910	"
2006	39.500	4.300	"

Source: (i) UNAIDS and WHO, AIDS Epidemic Update: December 1998, Geneva
(ii)UNAIDS and WHO, AIDS Epidemic Update: December 2005
(iii)WHO and UNAIDS, Global summary of the HIV and AIDS epidemic, 2006

Table 3

x	$1/A(x) = Y_x$	$\log_e(\frac{L}{A(x)} - 1)$	Total estimated HIV/AIDS Infected population (million)
0	.02994	-.054892578	33.6
1	.030182	-.043428193	34.383
2	.028657	-.147120949	35.164
3	.028028	-.19155194	35.941
4	.027388	-.247659016	36.715
5	.026693	-.307242356	37.484
6	.025429	-.425719712	38.246
7	.024824	-.487856416	39.003
8	.025316	-.4369965	39.75
		Total = -2.342465379	

$$\overline{X}(x) = .027649, \qquad \overline{Y}(x) = .0270271,$$

$$\sum_{1}^{n-1}(Y(x) - \overline{Y})^2 = 24.296X10^{-6} \quad \sum_{1}^{n-1}(X(x) - \overline{X})^2 = 26.758X10^{-6} \qquad b = .9529$$

Therefore

$$\mu = .048256$$

$$a = .007246 \qquad L = 65.016 \qquad \lambda^\wedge = -1.393608$$

The logistic curve is

$$A(x) = \frac{65.016}{1 + e^{.048256(-1.393608 - x)}}$$

The plot between the total estimated HIV/AIDS infected population and the parameter x is plotted in Fig.3

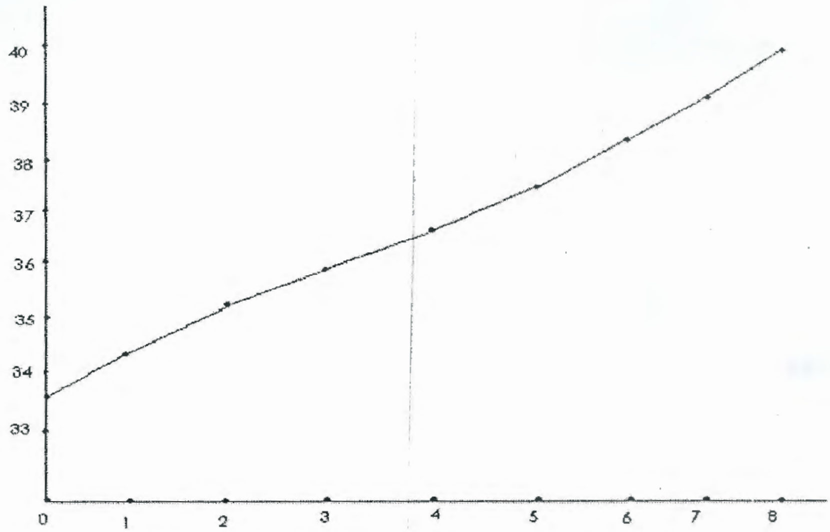

Figure 3. x vs. total estimated HIV/AIDS infected population

3. Conclusion

The hypothesis proposed in this paper follow the Logistic curve but fitting of this Logistic curve (methods of Rhodes), Fig.3, is neither exponential nor purely S-curve (biological shape), however, from population dynamic theory there are two models of population growth: Exponential curve (also known as a J-curve) which occurs when there is no limit to population size, and the Logistic curve (S-curve) shows the effect of a limiting factor. This shows that in assumed parameter x , which is function of both (α) and (β) , the increasing factors (α) of HIV/AIDS infected population is more dominant than the retarding factor (β) .Therefore, despite world wide efforts, the population of HIV/AIDS infected persons are still increasing with very high rate compared to the efforts which is being made to retard the growth of HIV/AIDS infected persons. For conspicuous retardation the logistic curve must follow the S-curve , as the S-curve represents that there are some limiting factor which retard the growth of the HIV/AIDS affected populations.

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